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Chapter 7: Genesis of a "Diophantine equation" in Arabic mathematics.

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Abstract

We intend in this paper to outline one part of the number theory genesis relatively to the equation: *a square plus/minus a number equals a square* focusing on the Arabic mathematicians' works, like, for example, those of al-Khazin, Ibn al-Laith, al-Karaji, al-Baghdadi and al-Khallat. We will then briefly describe the occidental tradition relative to his subject threw the works of Fibonacci, Fermat, Frenecle, Euler, and others.

Keywords: Arabic mathematics, the number theory, Diophantus, al-Khazin, Ibn al-Laith, al-Karaji, al-Baghdadi, al-Khallaṭ, Fibonacci, Fermat, Euler, Frenecle.

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1. Introduction

Translated into Arabic language by Qustā Ibn Lūqā, Diophantus's *Arithmetica*, translated into Arabic language by Qustā ibn Lūqā al-Baʿlabakkī in the Xth century, opened a loophole for the mathematicians of Bagdad. Abū Kāmil is, according to our knowledge, was the first to profit from this legacy. He devoted a whole book to the Diophantine equations: *The Funny book on calculus* (*Ṭarā'īf al-ḥisāb*), in which he studied several problems of undetermined analysis known by the Diophantine problems. He classified them into three categories: those who have a unique solution, those who have more than one solution and those with no solution:

> Many <persons> among specialists and non-specialists asked me about problems of this type. I answered them giving for the same problem one solution, if it has only one, two or three or may be more for others. The solution may however not exist. Until I found a problem that I resolved and I found many solutions for it. [1]

After him, this subject will be tackled by al-Khāzin, al-Khuğandī, a Persian astronomer and mathematician who was active during the end of the tenth century in Rayy, and Ibn al-Laith who were able to trigger on new researches related to Diophantine equations, like for example the theorem of al-Khāzin-al-Khuğandī: «there is no cube which can be written the sum of two cubes» which we can translate in modern notations as follows:

For every three positive integers *x*, *y* and *z*, we can't have the equality:

$$x^3 + y^3 = z^3$$

Actually, al-Khuğandī tried to prove the following theorem: «the sum of two cubes cannot be a cube». According to al-Khāzin, the proof of al-Khuğandī is not complete. Al-Khāzin tried however to prove that: «the sum of two squares cannot be a square and a cube cannot be divided in the sum of two cubes. A square can however be divided into a sum of two squares».
One should also note here that this problem was not only a crucial subject in many researches in Arabic mathematics but also in the accidental researches related to theory of numbers, to be solved in its general form by Wiles (Fermat-Wiles theorem, 1993-95).
During the thirteenth century Ibn al-Khawwām al-Baġdadī carried on the work on the Diophantine problems. In his book *The Bahāiyya benefits in calculation* (*al-Fawāid al-Bahā'iyya fi-l ḥisāb*) he mainly focused on the impossible ones. Ibn Mālik al-Dimashqī says in one of his texts*:* «Ibn al-Khawwām, God blesses him, quotes that it is not possible to prove the existence of a solution for it and asserts that he did not establish a proof for it regarding its impossibility» [2].

On the other hand, Roshdi Rashed asserts that the Book *the base of rule in the origin of calculation* (*Asās al-qāʾida fī aṣl al-ḥisāb*) of Kamāl al-Dīn al-Fārisī, which is actually a commentary of Ibn al-Khawwām's one, is the greatest work of al-Fārisī. In this text, al-Fārisī deals with the impossible Diophantine problems reaching almost the same results discovered by Ibn al-Khawwām. [3]

Ibn Mālik al-Dimashqī, we have cited above, is also one of those who worked on Diophantine problems. He especially retaken in his book «*The complete succor by pen's calculation*» (*al-Isʾāf al-'atamm bi ḥisāb al-qalam*) the works of Ibn al-Khawwām. Finally we note that in the XVth century, Bahā' al-Dīn al-ʿĀmilī tackled one more time the same subject, with tiny differences, in his *Summary of calculation* (*Khulāṣat al-ḥisāb*).

2. The equation a square plus/minus a number equals a square

One of the equations studied by Arabic mathematicians like al-Khāzin (Xth century), al-Karağī (XI^{e} century) is the famous:

$$x^2 - a = y_2^2 \text{ and } x^2 + a = y_1^2$$

Fibonacci of Pisa seems to be the first to settle this tradition in Europe. After him, Fermat, Frenecle, Euler and Gennocci carried on researches about the same subject to find other results.

2.1. The contribution of al-Khazin (Xth century)

In the tenth century, Al-Khāzin wrote two epistles on the equation mentioned above. In the one devoted to the construction and the importance of right-angled triangles with sides equal to whole number, he declares that several issues related to this subject have emerged. Like, for example, how to know odd numbers which can be decomposed into two squares? And also the method to establish right-angled triangles using two or three successive numbers or using odd numbers.
About the purpose behind these methods related to the construction of right-angled triangles al-Khāzin says in a fragment of his text:

> <The purpose> is to find a squared number if we add to it or we subtract from it a number the result is a squared number. [4]

Then, he proved the existence of a solution to this problem using Euclidian geometry. He precisely used in his demonstration the seventh proposition of the *Elements* Book. II

The proof of al-Khāzin: Assume that we have a right-angled triangle with sides x, y and z verifying:
$x^2 + y^2 = z^2$, then we have:

$z^2 + 2xy = (x+y)^2$ and $z^2 - 2xy = (x-y)^2$

So the number added in the first equation and subtracted in the second one is

$n = 2xy$

To argue how he found the added-subtracted number *n*, Al-Khāzin says:

> If we take any two successive numbers and we multiply them the one by the other, we then multiply the result by their sum and divide what we have by their difference so what comes is the added-subtracted number. Then we multiply every one of them by itself, we take the half of the sum and we divide it by the difference of the two numbers what is found is the root of the number that when we add to it the added-subtracted number then the result has a root and if we subtract from it that number the rest has a root.

Using our modern notation, this fragment can be translated as follows:

The added-subtracted number is:

$$n = \frac{ab[a+b]}{a-b}$$

And we have:

$$\left(\frac{a^2+b^2}{2(a-b)}\right)^2 \pm \frac{ab[a+b]}{a-b} = \left(\frac{a+b}{2} \pm \frac{ab}{a-b}\right)^2$$

After that, al-Khāzin explains how to solve the equation

$$x^2 \pm n = y_i^2, i = \{1,2\}$$

He says that starting from two different numbers, we can construct a right-angled triangle with sides equal to integers. This means:

If a and b are two integers and $a > b$. Suppose that:

$$x = (a+b)(a-b)$$
$$y = 2ab$$
$$z_1 = (a+b)(a-b) + 2ab$$
$$z_2 = (a+b)(a-b) - 2ab$$

Then the triangle with sides x, y and z1 and the one with sides x, y and z2 are both right-angled. The hypotenuse is z1 or z2.
And the added-subtracted number is:

$$n = 4ab(a+b)(a-b)$$

However, in his second epistle, and after presenting several results al-Khāzin explains the main purpose behind writing it:

> After what we have introduced, we arrive to our purpose, which is to prove that if we have a number how we can get a squared number that if we add to it this number and we subtract it from it the result and the difference are both squared numbers. [5]

Proposition 1

> How we ask for a squared number that if we add to it a given number and we subtract it from it the sum and the different we get are both squared numbers.

Before he answers this question, al-Khāzin introduces first the following proposition:

Lemma

> Every even number which can be divided into two squared numbers then its half can be divided into two squared numbers and the half of its half and so on.

The proof: Suppose that $2a = b^2 + c^2$, then

$$a = \frac{b^2+c^2}{2} = \left(\frac{b+c}{2}\right)^2 + \left(\frac{b-c}{2}\right)^2$$

Proof of proposition 1: Al-Khāzin says that after analyses he comes to prove the existence of three numbers verifying the system of equations

$$x^2 + n = y^2 \text{ and } x^2 - n = z^2$$

He first remarks that $z > x > y$ and then he concludes that x^2 is a sum of two squares since its double is a sum of two squares (referring to the lemma above):

$$2x^2 = y^2 + z^2$$

$$x^2 = (\frac{z+y}{2})^2 + (\frac{z-y}{2})^2$$
$$n = 2\left(\frac{z-y}{2}\right)\left(\frac{z+y}{2}\right)$$
$$2n = z^2 - y^2$$

So n is an even number and we have:

$$\frac{n}{2} = \left(\frac{z+y}{2}\right)\left(\frac{z-y}{2}\right)$$

Proposition 2: The imposed number should be in the form $4m(2n+1)$. Otherwise, the problem is impossible.

Proof of proposition 2: First, al-Khāzin proved that if

$$a^2 = b^2 + c^2$$

then b and c cannot be odd numbers nor even-even numbers. So b and c are both even or one is even and the other is odd and in this case the number a is on the form:

$$a = 4m(2n+1)$$

Otherwise the problem is impossible.

The proof of the opposite sense: If we have a double even and odd (*zawj al-zawj w'al-fard*) number and we search a number when we add to it that number the result is a square and when we subtract from it that <same> number the rest is <also> a square.

He supposes the divisors t and s of the number $\frac{a}{2}$.
So we have:

$\frac{a}{2} = st$ and $s^2 + t^2$ is a square. It means:

$x^2 = s^2 + t^2$

Al-Khāzin calls s and t the two peers. He then concluded that the first number of this category is 24 because its half is 12 and it is the inferior one of double-even and odd numbers. He takes all the numbers coming after 12 verifying the property: the sum of their squares equals a square and their product equals 12. So we find the two numbers 3 and 4. He says:

> The test/sign for the two sides of the supposed number <if you want to know> if the sum of their two squares is a square or not, is two divide the square of the smallest by the double of the greatest, so if the result has a root what we search is easy otherwise it is impossible.

Interpretation in modern symbols: If we part from an even number a verifying:

$\frac{a}{2} = s.t$ with $t > s$
and we want to know if

$s^2 + t^2 = square \longrightarrow s^2 + t^2 = u^2$

Then, we divide s^2 by $2t$ and the rest of the division is a square:

$$s^2 = 2tq + q^2$$
$$t^2 + s^2 = t^2 + 2tq + q^2 = (t+q)^2$$

Otherwise, i.e. when we don't have the condition:

$$s^2 = 2tq + q^2$$

It's difficult to resolve the equation

$$x^2 \pm a = y_i^2, i = \{1,2\},$$

or even impossible to do it.

Note: Arabic mathematicians classified this problem as a Diophantine one because it's a particular case of the example meet in Book III, Problem 23, of *The Arithmetica* that we can write with our notations as follows:

$$(x_1 + x_2 + x_3 + x_4)^2 \pm x_5 = y^2$$

2.2. The contribution of an unknown author

In an epistle whose author and tile are unknown, we find many new results related to the same subject and that seem to be an investigation of al-Khāzin's text mentioned above. In this fragment he gives a property of Pythagorean triplet with whole numbers:

> Know that all of these right triangles whose sides are expressible (*munṭaqa*) numbers, when the hypotenuse of one of them is you multiplied par itself and added to double of the two right sides product the result is a number with an expressible root, and when we subtract the double of the two right sides product from the hypotenuse, multiplied by itself, the result is <also> a number whose root is expressible.

The unknown author mentions, similarly to al-Khāzin, that he have drawn inspiration from Book II of Euclid's Elements. In this Book II we find the two remarkable identities demonstrated geometrically.

Interpretation in modern symbols: If we have a Pythagorean triplet (x, y, z), then $z^2 = x^2 + y^2$. So, $z^2 - 2xy = square$ and $z^2 + 2xy = square$

More precisely,

$$z^2 - 2xy = x^2 + y^2 - 2xy = (x - y)^2$$
$$z^2 + 2xy = x^2 + y^2 + 2xy = (x + y)^2$$

He describes later on the main purpose of his work:

> To repeat what it was proved by the ancients has no sense And this is the fundament of <the problem> a *māl* with a root if you add to it a known number what it comes have a root and if you subtract from it the same number what it rests has a root.

So the mathematical problem as it was highlighted by this author is:
Given a whole number a, how can we find three numbers x, $y_1$and y_2 verifying:

$x^2 + a = y_1^2$ and $x^2 - a = y_2^2$?

One of the important results established by this author is the following:
If we have a Pythagorean triplet (x, y, z) then,

$$z^2 = x^2 + y^2$$

This leads to:

$$z^2 - 2xy = (x-y)^2 = {z_1}^2$$
$$z^2 + 2xy = (x+y)^2 = {z_2}^2$$

So: necessarily

$${z_1}^2 \equiv 1[10] \text{ or } {z_2}^2 \equiv 9[10]$$
$${z_1}^2 \not\equiv 5[10] \text{ and } {z_2}^2 \not\equiv 5[10]$$

On the other side, and since every Pythagorean triplet can be in the form:

$$(a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2$$

Consequently, we get a primary right-angled triangle if we take two numbers a and b verifying: $a^2 + b^2$ is an odd number. Hence, a and b are of two opposite parities. Then, $a + b$ is also odd. This means that $a + b = 2n + 1$. For this reason, to search a right triangle is to search how to decompose an odd number.

Note:

Suppose that:

$$2n + 1 = n + n + 1 = (n + 1 + c) + (n - c)$$
$$c \in \{0,1,2\ldots,n-1\}$$
$$a^2 - b^2 = (n + c + 1)^2 - (n - c)^2 = (2n + 1)(2c + 1)$$
$$a^2 + b^2 = (n + c + 1)^2 + (n - c)^2 = 2n(n + 1) + 2c(c + 1) + 1$$
$$2ab = 2(n + c + 1)(n - c)$$

So, according taking in our account such a decomposition, the two numbers $n + c + 1$and $n - c$ should verify:

$(n + c + 1) \wedge (n - c) = 1.$

And so:

$$x = (n + c + 1)^2 - (n - c)^2 \quad y = 2(n + c + 1)(n - c)$$
$$z_1 = x + y = (2n + 1)^2 - 2(n - c)^2$$
$$z_2 = x - y = (2c + 1)^2 - 2(n - c)^2$$

And then: z_1 and z_2 are both odds. And so their squares are in the form:

$$10k + 1, 10k + 5, 10k + 9$$

However, the unknown author proved that they cannot be in the form $10k + 5$. As a consequence they can only be from the two categories:

$$10k + 1 \text{ or } 10k + 9$$

Our proof of the second proposition: (in modern notations): Actually the Author did not give the proof of this issue. We give here, in modern notation an eventual one.

If $z_1^2 \equiv 5[10]$ and $z_2^2 \equiv 5[10]$ then z_1^2 and z_2^2 are divisible by 5. But: $z_1$and z_2 can be written in the form:

$\mathrm{f}^2 - 2\mathrm{g}^2$

But, we have:

$\mathrm{f}^2 \in \{5k; 5k + 1; 5k + 4\}$

Then, we should have:

$2g^2 \in \{5k; 5k+2; 5k+3\}$

As a consequence: $f^2 - 2g^2$ is divisible by 5 if and only if: f and g are both divisible by 5 and then, $2n+1$ and $n-c$ are divisible by 5, as the same for $2c+1$ and $n-c$.
Similarly: $(n+1)-(n-2c)$ and $(c+1)+(n-2c)$ are divisible by 5 if $n+c+1$ and $n-c$ are divisible by 5. And as a consequence the right triangle is not a primary one.

Afterwards, the author gives some examples:

Example: If we part from the primary right-angled triangle: $(3,4,5)$ then:
$5^2 + 2 \times 3 \times 4 = 49 = 7^2 = (3+4)^2 \equiv 9[10]$
$5^2 - 2 \times 3 \times 4 = 1 = 1^2 = (4-3)^2 \equiv 1[10]$
He afterwards gives an array in which he shows that the hypotenuse 65 appears twice in two triangles and takes the place of 49 which is abandoned.

(x,y,z)	$x^2 + 2yz = y_1{}^2$	$x^2 - 2yz = y_2{}^2$
(17,8,5)	$17^2 + 2 \times 8 \times 15 = 23^2$	$17^2 - 2 \times 8 \times 15 = 7^2$
(25,24,7)	$25^2 + 2 \times 24 \times 7 = 31^2$	$25^2 - 2 \times 24 \times 7 = 17^2$
(29,20,21)	$29^2 + 2 \times 20 \times 21 = 41^2$	$29^2 - 2 \times 20 \times 21 =$
	$37^2 + 2 \times 12 \times 35 = 47^2$	$37^2 - 2 \times 12 \times 35 = 23^2$
(41,40,9)	$41^2 + 2 \times 40 \times 9 = 49^2$	$41^2 - 2 \times 40 \times 9 = 31^2$

He says that: the hypotenuse in the fourteenth rank is 49 and it cannot be a hypotenuse of a right triangle with two sides equal to two whole numbers because it cannot be split into two perfect squares.

(53,28,45)	$53^2 + 2 \times 28 \times 45 = 73^2$	$53^2 - 2 \times 28 \times 45 = 17^2$
(61,60,11)	$61^2 + 2 \times 60 \times 11 = 71^2$	$61^2 - 2 \times 60 \times 11 = 49^2$
(65,16,63)	$65^2 + 2 \times 16 \times 63 = 79^2$	$65^2 - 2 \times 16 \times 63 = 47^2$
(65,56, 33)	$65^2 + 2 \times 56 \times 33 = 89^2$	$65^2 - 2 \times 56 \times 33 = 23^2$

He afterwards gives an array in which he shows that the hypotenuse 65 appears twice in two triangles and takes the place of 49 which is abandoned.

2.3. The contribution of al-Karaji

Arabic mathematicians of the Xth century pursued investigating the same subject related to the right-angled triangle sides. One of them is Abū al-Ḥusayn al-Karağī, the father of polynomials.
In his text *al-Badī'*, al-Karağī studied the equation:

$x^2 + 5 = y_1^2$ and $x^2 - 5 = y_2^2$

> To infer it we take the difference between one *māl* plus five units and one *māl* mines five units, which is ten. Then, you divide it by a quantity if you add it to the quotient and you take the square of the half of you get and you subtract five from it the rest is a square. What you need to perform it is *Istiqrā*. [6]

Thanks to that al-Karağī gets

$$x^2 = \frac{1681}{144}$$

2.4. The contribution of al-Baghdādī (980-1037) a new method

In his book *al-Takmila fi-l ḥisāb*, al-Baghdādī [7] deals with the same problem saying:

> What can be known only by cubes is the extraction of a *māl* <of a whole number> if you add to it a number the result is a perfect square (*muğḏḏar*) and if you subtract the same number from it the result is a perfect square. You preserve the root of the result/difference, you add to it the square of the cubic root of that cube plus one and you divide the result by the preserved root what it comes is the root of the searched *māl*.

The method of al-Baghdādī: This method used by al-Baghdādī can be written in our notations as follows:

To choose the suable number *a* verifying:

$x^2 \pm a = y_i^2$,

we should manage things in order to get:

$$\frac{x^3 - x}{a} = square$$

Suppose that:

$$\frac{x^3 - x}{a} = y^2$$

Then, the added-subtracted number a should be:

$$a = \frac{x^2 + 1}{2y}$$

Afterwards al-Baghdādī uses his method to resolve the same example given by al-Karağī, we mean.

$x^2 + 5 = y_1^2$ and $x^2 - 5 = y_2^2$

He performs the operation: $\frac{9^3-9}{5} = 144 = 12^2$, and says that the demanded number is so:

$$\frac{9^2 + 1}{2 \times 12} = \frac{41}{12}$$

2.5. The contribution of al-Zanjani (Xth century)

During the end of the XIII[th] century, the same subject was studied by al-Zanğānī. [8]
He presented the problem and how to solve it saying

> If he says: a square if we add to it a number we get a square and if we subtract fro it the same number we get a square. Know first that every two numbers if they are multiplied the one by the other twice and it's what we call the two complementary (*al-mutammimayn*), if its added to the sum of their squares it gives a square whose root is the sum of the two numbers and if its subtracted from the sum of their squares the rest is a square whose root is the difference between the two numbers.

Note: In this fragment al-Zanğānī says:
If we have two numbers *a* and *b*, so:

The added-subtracted number is: $2ab$
The great square is: $a^2 + b^2 + 2ab = (a + b)^2$
The small square is: $a^2 + b^2 - 2ab = (a - b)^2$

Al-Zanǧānī specifies, and proves it in his book *'Omdat Al-ḥisāb* by *al-Istiqrā*, that the added-subtracted number can never be a square.
We mention here that this theorem was attributed by historians of mathematics to Fibonacci.

2.6. The contribution of Ibn al-Ha'im

In his text *Al-ma'ūna* (The help) Ibn al-Hā'im studied the equation:

$x^2 + 5 = y_1^2$ and $x^2 - 5 = y_2^2$

He says using the same technique of al-Baghdādī threw an example:

> If it is said: a square if you add five to it the sum is a square and if you subtract five from it the rest is a square? So demand a cube if you subtract from it its cubic root and the rest is divided by five, then the quotient is a square. You find it seven hundred and twenty nine. Subtract from it its cubic root which is nine and divide the rest by five the quotient is one hundred and forty four and its root is twelve. Retain it and add one, as al-Karaǧī did in al-*Badī'*, to the square of its cubic root which is eighty- one and divide the half of the result which is fourth one by the retained twelve, it comes three and one quarter and one sixth.

2.7. The contribution of Ibn al-Khawwam

In his book *Fawāid al-Bahāiyya fi-l ḥisāb* Ibn al-Khawwām have mentions a list of 33 impossible equations among them the equation:

$x^2 \pm 10 = y_i^2, i = \{1,2\}$

2.8. The contribution of al-Dimashqui

Ibn Mālik al-Dimashqī just retakes the unsolved problems in the text of Ibn al-Khawwām and regroup them in his book in one chapter: A square if we add to it or we subtract from it ten the two results are both squares. [9]

3. Conclusion

The numbers theory represents a crucial subject in mathematics. Researches on this topic are undertaken now with the help of new technologies. However, the beginnings were nor easy neither evident. Arabic mathematicians had well participated in the development of such a tradition. Actually, taking advantage of the translations of the Latin scientific legacy, the Successors of al-Ḫwārizmī, who has the merit to establish a new discipline: algebra, revisit the Euclid's *Elements* and the *Arithmetica* of Diophantus using al-Ḫwārizmī's lexicon and methods. Henceforth, several new topics and chapters have emerged. The pursuit of the works undertaken by the Pythagorean School on the numbers characteristics was therefore initiated in Bagdad with more abstraction.
As an example, inspired by a Diophantine equation, Arabic mathematicians, revisited the right-angled triangle sides' property, known by the Pythagorean Theorem and proposes a new system of equations that we can write in our modern notations as follows:

$x^2 + a = {y_1}^2$ and $x^2 - a = {y_2}^2$

Which is new in their issue is that they stipulate for the solution, if the equation has one, to be a whole number, i.e. integers and not fractions.

Al-Khāzin is one of those who tackled this subject. In the Xth century, he devoted two epistles in which he gives firstly proved the existence of a solution starting from a Pythagorean triplet. Conversely, he shows how to get a right angled triangle starting from two given numbers. More than that, he stipulate in the second epistle for the added subtracted number a to be a double even odd number, otherwise the problem has no solution, In our modern notation this can be written as follows: $a = 4m(2n + 1)$.

The same subject interested an unknown author of the Xth century. In his text he gives conditions on the terms ${y_1}^2$ and ${y_2}^2$: the rest of their division by ten should be one or nine and never 5. Another original result of this author is a list of the eventual values of the primary added-subtracted number a.

After these two initiatives, al-Karağī, an engineer, algebraist and arithmetician propose to resolve the problem with a number a equals to five i.e. regardless the condition stipulated by al-Khāzin. The solution given by al-Karağī for the equation

$x^2 + 5 = {y_1}^2$ and $x^2 - 5 = {y_2}^2$

is the fraction:

$$x = \frac{41}{12}$$

Almost in the same period al-Baghdadi involves indeed algebra to solve such an arithmetical problem, by searching a solution by the mean of cubes. In the thirteenth century, al-Zanǧānī assured that the added-subtracted number can never be a square. The same assertion we find it in a text of Fibonacci, the Italian mathematician: «Nullus quadratus numerus potest esse conguum».

One of the consequences of the result settled by al-Zanǧānī is that: «The surface of a right-angled (With whole numbers sides) triangle can never be a square», which is a result that Fermat have quoted in the XVIth century and after him Frenicle in XVII one.

To finish, we may note that another important result, which represents a special case of the famous Fermat Theorem, was proved by al-Khayyam in the XIth century is:

« The equation $x^4 + y^4 = z^4$ has no integer solutions». This same problem will be revisited in Europe by Fermat, Legendre and Euler.

What we have studied in this work, is actually a tiny part of the contribution of Arabic mathematicians in the development of numbers theory. We intend, in a future work, to highlight others works belonging to Arabic tradition in relation with this subject and secondly to focus on European mathematicians contributions in order to bring out an eventual relationship between their works and those of Arabic Works.

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