Semigroup’s approach to the study of the Hölder continuous regularity for Laplace equation in nonsmooth domain.

Belkacem Chaouchi a *,

a Laboratoire de l’Energie et des Systèmes Intelligents, Khemis Miliana University, 44225, Algeria.

ARTICLE INFO

Article history :
Received December 2013
Accepted April 2014

Keywords :
Fractional powers of linear operators ;
Analytic semigroup ;
Operational differential equation of elliptic type ;
Little Hölder space ;
Cuspidal point ;
Interpolation spaces.

ABSTRACT

We will investigate the Dirichlet problem for Laplace equation set in a singular domain with cuspidal point. We look to describe the behavior of the unique solution near the cuspidal point in the framework of the little Hölder space \( h^{2\sigma} (\Omega) \) with \( \sigma \in [0, 1/2] \).

©2014 LESI. All right reserved.

1. Introduction

Let \( \Omega \subset \mathbb{R}^2 \) a bounded domain. We assume that its boundary \( \partial \Omega \) is of class \( C^\infty \) except at the origin \((0, 0)\) where \( \partial \Omega \) has a cuspidal point. To be more precise, assume that we can choose cartesian coordinates so that

\[
\Omega_{x_0} := \Omega \cap B(0, x_0) = \left\{ (x, y) \in \mathbb{R}^2 : 0 < x < x_0, 0 < y < \psi(x) \right\},
\]

where \( B(0, x_0) \) is the ball of center 0 and radius \( x_0 \). Here \( x_0 > 0 \) is small enough and \( \psi \) is a real function satisfying the following conditions

1. \( \psi \in C^2 ([0, x_0]) \cap C^\infty ([0, x_0]) \).
2. \( \psi < 0 \) on \([0, x_0]\).
3. \( \int_0^{x_0} \frac{dt}{\psi(t)} \) diverges.
4. \( \psi(0) = \psi'(0) = 0 \).

*Email : chaouchicukm@gmail.com
5. \( \psi(0) \psi''(0) = 0 \).

6. We assume also that \( \psi \) can be extended to \([x_0, +\infty[\), so that \( \frac{1}{\psi} \) remains in \( L^1([x_0, +\infty[) \).

Consider the problem

\[ \Delta u = h, \quad \text{in } \Omega, \]  \hfill (1)

under the homogenous boundary conditions

\[ u = 0, \quad \text{on } \partial \Omega. \]  \hfill (2)

The right hand term \( h \) is taken in the little Hölder space \( h^{2\sigma}(\Omega) \) which denotes the subspace of \( C^{2\sigma}(\Omega) \) consisting of the functions \( f \) such that

\[ \forall (x, y), (x', y') \in \Omega : \lim_{\delta \to 0^+} \sup_{0 < \| (x, y) - (x', y') \| \leq \delta} \frac{\| f(x, y) - f(x', y') \|_E}{\| (x, y) - (x', y') \|^{2\sigma}} = 0. \]

We assume also that

\[ h(x_0, 0) = h(x_0, \psi(x_0)) = 0, \]  \hfill (3)

(more details about these spaces will be given later).

The study of elliptic problems posed in cusp domains was considered by numerous authors, see e.g. [7] and [12] (and the extensive bibliography therein). The majority of these works deals with the \( L^p \) setting of these problems. Comparatively, there are a few results concerning the little Hölder regularity. The difficulty related to this kind of problems comes from the typical properties of the functional framework. In our situation, the classical methods such as the variational method do not apply, see [9], [10]. Furthermore, we know that such spaces have not the UMD character, see [3]. This explains why the operational approach used in [1], [6], [7] is excluded. Hence, we use an alternative approach, namely the theory of abstract differential equations. This technique has been fruitfully used in [4] to prove some regularity results for Problem (1). These results are restricted to the domain

\[ \Omega = \{ (x, y) \in \mathbb{R}^2 : 0 < x < x_0, \ - x^\alpha < y < x^\alpha \} , \]

with \( 1 < \alpha \leq 2 \).

In this work, we follow the same strategy. Our goal is to give a complete study of the problem (1) in the neighborhood of the cusp edge it means in \( \Omega_{x_0} \). We will prove that Problem (1)-(2) has a unique strict solution. Furthermore, we show that the regularity of this solution near the cuspidal edge is dependent to the geometry of the domain \( \Omega_{x_0} \) and the exponent \( \sigma \).

The schedule of the paper is the following one: In Section 2, we introduce some notations and definitions concerning some functions spaces to be used throughout this paper. Section 3, there are two main steps. First, we use an appropriate change of variables to transform our singular domain \( \Omega \) into a fixed one. Secondly, we write the transformed problem as an abstract differential equation of elliptic type. Section 4, is devoted to the complete study of the abstract version of the transformed problem, the techniques are essentially based on the use of the semigroup’s theory and some interpolation spaces. Section 5, we go back to the original domain and we give our main result describing the regularity of the solution \( u \) of Problem (1)-(2).
2. Preliminaries

In this work, it is necessary to introduce some Banach spaces of vector-valued functions. Let \((E, \|\cdot\|_E)\) be a complex Banach space and let \(\mu \in [0, 1]\). We consider the following functional spaces \(B ([0, +\infty[, E) , C ([0, +\infty[, E) , C^2 ([0, +\infty[, E)\) consisting respectively of the bounded, continuous, 2 times continuously differentiable functions \(f : [0, +\infty[ \to E\).

We set also
\[
C_b ([0, +\infty[, E) = \left\{ f \in C ([0, +\infty[, E) : \lim_{\xi \to +\infty} f (\xi) = 0 \right\},
\]
\[
C^2_b ([0, +\infty[, E) = C^2 ([0, +\infty[, E) \cap C_b ([0, +\infty[, E).
\]

The Banach spaces of Hölder continuous functions \(C^\mu ([0, +\infty[, E)\) is defined by
\[
C^\mu_b ([0, +\infty[, E) = \left\{ f \in C_b ([0, +\infty[, E) : \sup_{\xi_1, \xi_0 \geq 0} \frac{\|f (\xi_1) - f (\xi_0)\|_E}{|\xi_1 - \xi_0|^{\mu}} < +\infty \right\}, \text{with}
\]
\[
\|f\|_{C^\mu_b ([0, +\infty[, E)} := \|f\|_{C([0, +\infty[, E)} + \sup_{\xi_1, \xi_0 \geq 0} \frac{\|f (\xi_1) - f (\xi_0)\|_E}{|\xi_1 - \xi_0|^{\mu}}.
\]

The Banach spaces of little Hölder continuous functions \(h^\mu ([0, +\infty[, E) , h^{\mu+2} ([0, +\infty[, E)\) are defined by
\[
h^\mu_b ([0, +\infty[, E) = \left\{ f \in C^\mu_b ([0, +\infty[, E) : \lim_{\delta \to 0} \sup_{\xi_1, \xi_0 \geq 0, |\xi_1 - \xi_0| \leq \delta} \frac{\|f (\xi_1) - f (\xi_0)\|_E}{(\xi_1 - \xi_0)^\mu} = 0 \right\}.
\]
\[
h^{\mu+2}_b ([0, +\infty[, E) = \{ f \in C^2_b ([0, +\infty[, E) : f , f' , f'' \in h^\mu ([0, +\infty[, E) \}\).
\]
\[
L^\infty ([0, +\infty[, E) = \left\{ f : [0, +\infty[ \to E , \text{ Bochner measurable and } \sup_{\xi \in U} \text{ess } \|f (\xi)\|_E < \infty \right\}.
\]

Remark 1 For \(\mu \in [0, 1]\), One has
\[
C^2_b ([0, +\infty[, E) \subset C^1_b ([0, +\infty[, E) \subset h^\mu_b ([0, +\infty[, E) \subset C^\mu_b ([0, +\infty[, E) \subset C_b ([0, +\infty[, E).
\]

Remark 2 For \(\mu \in [0, 1]\), every function of \(h^\mu (\Omega)\) can be extended to a function of \(h^\mu (\overline{\Omega})\).

3. Change of variables

We set
\[
\Pi : \Omega_{x_0} \to Q_{\xi_0}
\]
\[
(x, y) \mapsto \left( \xi := \theta^{-1} (x) := - \frac{+\infty}{x} \int \frac{d\sigma}{\psi (\sigma)} , \eta := \frac{y}{\psi} \right),
\]
where \(Q_{\xi_0}\) is the semi-infinite strip
\[
Q_{\xi_0} = [\xi_0, +\infty] \times [0, 1], \ \xi_0 = - \int_{x_0}^{+\infty} \frac{d\nu}{\psi (\nu)},
\]

32
which mean that the cuspidal point \((0, 0)\) is transformed in
\[
D_{\infty} = \{(+\infty, \eta) : \eta \in [0, 1]\} = \{+\infty\} \times [0, 1].
\]  

Now, consider the following change of functions
\[
\begin{align*}
v (\xi, \eta) & := u (x, y), \\
g (\xi, \eta) & := h (x, y).
\end{align*}
\]  

Consequently. Problem \((1)-(2)\) is equivalent to
\[
\begin{align*}
\Delta_{(\xi, \eta)} v (\xi, \eta) + [Lv] (\xi, \eta) & = f (\xi, \eta), \quad (\xi, \eta) \in Q_{\xi_0}, \\
v (\xi_0, \eta) & = 0, \quad 0 < \eta < 1, \\
v (\xi, 0) = v (\xi, 1) & = 0, \quad \xi > \xi_0,
\end{align*}
\]  

where
\[
f (\xi, \eta) = \psi^2 g (\xi, \eta),
\]  

and \(L\) is the second differential operator with \(C^\infty\)-bounded coefficients on \(Q_{\xi_0}\) given by
\[
[Lv] (\xi, \eta) = \eta^2 (\psi')^2 \partial^2_\eta v (\xi, \eta) + 2\eta \psi' \partial^2_{\eta \xi} (\xi, \eta) \\
+ \psi' \partial \xi v (\xi, \eta) - \eta \left(2 (\psi')^2 - (\psi \psi'')^2\right) \partial_\eta v.
\]  

**Remark 3** Note here that
\[
\forall \eta \in [0, 1] : \lim_{\xi \to +\infty} g (\xi, \eta) = \lim_{\xi \to +\infty} h (\theta (\xi), \eta \psi (\theta (\xi))) = h (0, 0).
\]  

and
\[
\forall \eta \in [0, 1] : \lim_{\xi \to +\infty} f (\xi, \eta) = \lim_{\xi \to +\infty} \psi^2 (\theta (\xi)) h (\theta (\xi), \eta \psi (\theta (\xi))) = 0.
\]  

In the sequel, we will focus ourselves on the study of the concrete problem :
\[
\begin{align*}
\Delta_{(\xi, \eta)} v (\xi, \eta) & = f (\xi, \eta), \quad (\xi, \eta) \in Q_{\xi_0}, \\
v (\xi_0, \eta) & = 0, \quad 0 < \eta < 1, \\
v (\xi, 0) = v (\xi, 1) & = 0, \quad \xi > \xi_0.
\end{align*}
\]  

Using the same argument as in [4] and [8], one has

**Proposition 4** Let \(\sigma \in \left]0, \frac{1}{2}\right[\). Then
\[
h \in h^{2\sigma} (\overline{Q}_{x_0}) \Rightarrow f \in h^{2\sigma} (\overline{Q}_{\xi_0}).
\]
3.1. The abstract formulation of the problem (10)

Set $E = C([0, 1])$ endowed with its usual norm. Define the vector-valued following functions:

$v : [\xi_0, +\infty[ \to E; \xi \mapsto v(\xi); \quad v(\xi)(\eta) = v(\xi, \eta),$

$f : [\xi_0, +\infty[ \to E; \xi \mapsto f(\xi); \quad f(\xi)(\eta) = f(\xi, \eta).$

Consider the operator $A$ defined by

\[
\begin{aligned}
D(A) &= \{ w \in C^2([0, 1]) : w(0) = w(1) = 0 \},
\quad \text{and} \\
(Aw)(\eta) &= D_\xi^2 w(\eta).
\end{aligned}
\] (11)

Then, the concrete problem

\[
\begin{aligned}
v''(\xi) + Av(\xi) &= f(\xi), \quad \xi > \xi_0, \\
v(\xi_0) &= 0, \quad 0 < \eta < 1,
\end{aligned}
\] (12)

is written in the following operational form

\[
\begin{aligned}
v''(\xi) + Av(\xi) &= f(\xi), \quad \xi > \xi_0, \\
v(\xi_0) &= 0.
\end{aligned}
\] (13)

In order to obtain more optimal results for the problem (13), it will be more convenient to study the problem

\[
\begin{aligned}
v''(\xi) + Av(\xi) &= f(\xi), \quad \xi > \xi_0, \\
v(\xi_0) &= \varphi,
\end{aligned}
\] (14)

where

$f \in h_b^{2\sigma}(Q_{\xi_0}),$

and $\varphi \in E.$

It is well known that $A$ it is a closed non densely defined operator satisfying the Krein-ellipticity property, that is : $\mathbb{R}^+ \subset \rho(A)$ and

\[
\exists \ C > 0 : \forall \lambda \geq 0 \quad \|(A - \lambda I)^{-1}\|_{L(E)} \leq \frac{C}{1 + |\lambda|},
\] (16)

(here $\rho(A)$ is the resolvent set of $A$).

Assumption (16) implies that operator $B = -(-A)^{1/2}$ is well defined and it is the infinitesimal generator of the generalized analytic semigroup $(e^{tB})_{t \geq 0}.$ More precisely, there exists a sector

$\Pi_{\delta, r_0} = \{ z \in \mathbb{C}^* : \arg z \leq \delta + \pi/2 \} \cup B(0, r_0),$

(with some positive $\delta, r_0$) and $C > 0$ such that $\rho(B) \supset \Pi_{\delta, r_0}$ and

\[
\exists C > 0 : \forall z \in \Pi_{\delta, r_0}, \quad \|(B - z I)^{-1}\| \leq \frac{C}{1 + |z|}.
\]
Remark 5 Let us state some properties of the analytic semigroup \((e^{\xi B})_{\xi > 0}\):

1. \(\exists \omega > 0 : \forall k \in \mathbb{N}, \exists m_k \geq 1\) such that
   \[
   \|\xi^k B^k e^{\xi B}\|_{L(E)} \leq m_k e^{-\omega \xi},
   \]  
   (17)

2. \(\lim_{\xi \to 0} e^{\xi B} \varphi = \varphi\) if and only if \(\varphi \in D(B)\).

Note that in our case, one has
\[
D(A) = \{\psi \in C([0, 1]) : \psi(0) = \psi(1) = 0\} = D(B),
\]
Thanks to Assumption (16), we introduce the real Banach interpolation spaces between \(D(A)\) and \(E\):
\[
D_A(2\sigma) = \left\{ \zeta \in E : \lim_{r \to +\infty} \|r^{2\sigma} A (A - rI)^{-1} \zeta\|_E = 0 \right\}, \sigma \in ]0, 1/2[.
\]
More details about these Banach spaces are given in [11] and [14].

As in [2] and [13], we are studying our equation (14) under the hypothesis
\[
f \in L^\infty([\xi_0, +\infty[; h^{2\sigma}([0, 1])) \cap h^{2\sigma}_0([\xi_0, +\infty[; C([0, 1])).
\]  
(18)

4. Optimal results for Problem (14)

Consider the natural change of function: for \(\xi \in [0, +\infty[\), set
\[
V(\xi) = v(\xi + \xi_0), \quad F(\xi) = f(\xi + \xi_0),
\]
where \(v\) is the eventual solution of (14) and \(f\) satisfying (18). Therefore it is clear that
\[
F \in L^\infty([0, +\infty[; h^{2\sigma}([0, 1])) \cap h^{2\sigma}_0([0, +\infty[; C([0, 1]));
\]  
(19)

now the complete analysis of (14) on \([\xi_0, +\infty[\) is equivalent to the one done for the following
\[
\begin{cases}
V''(\xi) + AV(\xi) = F(\xi) & \xi > 0, \\
V(0) = \varphi,
\end{cases}
\]  
(20)

on \([0, +\infty[\).

Let us focus ourselves on the study of the problem (20). The solution of (20) is given formally by
\[
V(\xi) = e^{B\xi} \varphi + \frac{1}{2} \int_0^{+\infty} e^{B(\xi+s)} B^{-1} F(s) ds
\]  
(21)

\[
-\frac{1}{2} \int_0^{\xi} e^{B(\xi-s)} B^{-1} F(s) ds
\]

\[
-\frac{1}{2} \int_{\xi}^{+\infty} e^{B(s-\xi)} B^{-1} F(s) ds.
\]
Note that the absolute convergence of the second and fourth integral is obtained due to the estimate (17). In fact, for instance, one has

\[
\left\| \int_{\xi}^{+\infty} e^{B(s-\xi)} B^{-1} F(s) ds \right\| \leq C \left( \int_{\xi}^{+\infty} e^{-\omega(s-\xi)} ds \right) \max_{t \in [0, +\infty]} \| F(t) \|_E
\]

\[
\leq C \max_{t \in [0, +\infty]} \| F(t) \|_E .
\]

We can state some regularity properties of \( V \):

**4.1. Optimal results using** \( F \in h^2_B ([0, +\infty]; E) \)

In this subsection we will use the fact that \( F \in h^2_B ([0, +\infty]; E) \);

recall also that \( F \) verifies

\[
\lim_{\xi \to +\infty} F(\xi) = 0 .
\]  

(22)

Assume that \( \varphi \in D (A) = D (B^2) \). Write

\[
V(\xi) = e^{B\xi} \varphi + \frac{1}{2} \int_{0}^{+\infty} e^{B(\xi+s)} B^{-1} F(s) ds
\]

\[
- \frac{1}{2} \int_{0}^{\xi} e^{B(\xi-s)} B^{-1} F(s) ds
\]

\[
- \frac{1}{2} \int_{\xi}^{+\infty} e^{B(s-\xi)} B^{-1} F(s) ds
\]

\[
= V_1(\xi) + V_2(\xi) + V_3(\xi) + V_4(\xi).
\]

Then

\[
\| B^2 V_1(\xi) \|_E = \| e^{B\xi} B^2 \varphi \|_E \leq C \| \varphi \|_{D(B^2)} .
\]

Concerning \( V_2(\xi) \), one has

\[
V_2(\xi) = \frac{B^{-1}}{2} \int_{0}^{+\infty} e^{B(\xi+s)} F(s) ds \in D (B) ,
\]

and

\[
BV_2(\xi) = \frac{e^{B\xi}}{2} \int_{0}^{+\infty} e^{Bs} F(s) ds
\]

\[
= \frac{e^{B\xi}}{2} \int_{0}^{+\infty} e^{Bs} (F(s) - F(0)) ds + \frac{1}{2} B^{-1} e^{B\xi} F(0),
\]

36
from which it follows that

\[
B^2 V_2 (\xi) = \frac{e^{B\xi}}{2} \int_0^{+\infty} B e^{Bs} (F(s) - F(0)) ds + \frac{1}{2} e^{B\xi} F(0),
\]

and clearly

\[
\| B^2 V_2 (\xi) \|_E \leq C \left( \int_0^{+\infty} e^{-\omega s} s^{2\sigma - 1} ds \right) \| F \|_{h^2_{\sigma}([0,+\infty];E)} + C' \| F(0) \|_E
\]

\[
\leq C \frac{\Gamma(2\sigma)}{\omega^{2\sigma}} \| F \|_{h^2_{\sigma}([0,+\infty];E)} + C' \| F(0) \|_E
\]

\[
\leq C \| F \|_{h^2_{\sigma}([0,+\infty];E)},
\]

where \( \Gamma \) is the usual Euler function defined by

\[
\Gamma(z) = \int_0^{+\infty} e^{-w} w^{z-1} dw, \quad \text{Re} \, z > 0.
\]

For \( V_3(\xi) \), by writing

\[
BV_3 (\xi) = -\frac{1}{2} \int_0^{\xi} e^{B(\xi-s)} F(s) ds
\]

\[
= -\frac{1}{2} \int_0^{\xi} e^{B(\xi-s)} (F(s) - F(\xi)) ds - \frac{B^{-1}}{2} e^{B\xi} F(\xi) + \frac{B^{-1}}{2} F(\xi);
\]

we have

\[
B^2 V_3 (\xi) = -\frac{1}{2} \int_0^{\xi} B e^{B(\xi-s)} (F(s) - F(\xi)) ds - \frac{1}{2} e^{B\xi} F(\xi) + \frac{1}{2} F(\xi);
\]

thus

\[
\| B^2 V_3 (\xi) \|_E \leq C \| F \|_{h^2_{\sigma}([0,+\infty];E)} + C' \| F(\xi) \|_E \leq C'' \| F \|_{h^2_{\sigma}([0,+\infty];E)}.
\]

Finally

\[
BV_4 (\xi)
\]

\[
= -\frac{1}{2} \int_0^{+\infty} e^{B(s-\xi)} F(s) ds
\]

\[
= -\frac{1}{2} \int_0^{+\infty} e^{B(s-\xi)} (F(s) - F(\xi)) ds + \frac{B^{-1}}{2} F(\xi),
\]

37
and

\[ B^2 V_4(\xi) = -\frac{1}{2} \int_{\xi}^{+\infty} B e^{B(s-\xi)} (F(s) - F(\xi)) \, ds + \frac{1}{2} F(\xi), \]

which gives the estimate

\[ \| B^2 V_4(\xi) \|_E \leq C \| F \|_{h^{2\sigma}([0, +\infty[:E]} + C' \| F(\xi) \|_E \leq C'' \| F \|_{h^{2\sigma}([0, +\infty[:E)}. \]

Summarizing, we obtain the following decomposition

\[ B^2 V(\xi) = e^{B\xi} \left[ B^2 \varphi + F(0) \right] + F(\xi) \]

\[ + \frac{e^{B\xi}}{2} \left( \int_{0}^{+\infty} B e^{B s} (F(s) - F(0)) \, ds \right) \]

\[ - \frac{1}{2} \int_{0}^{\xi} B e^{B (\xi-s)} (F(s) - F(\xi)) \, ds \]

\[ - \frac{1}{2} \int_{\xi}^{+\infty} B e^{B (s-\xi)} (F(s) - F(\xi)) \, ds. \]

**Proposition 6** Let \( V \) given in (23). Then \( \lim_{\xi \to +\infty} V(\xi) = 0. \)

**Proof.** We know that there exists \( \omega > 0 \) and \( m_0 > 0 \) such that for any \( \xi > 0 \)

\[ \| V_1(\xi) \|_E \leq m_0 e^{-\omega \xi} \| \varphi \|_E, \]

then

\[ \lim_{\xi \to +\infty} \| V_1(\xi) \|_E = 0. \]

the same is true for \( V_2(\xi) \).

For \( V_3(\xi) \), one write

\[ V_2(\xi) \]

\[ = -\frac{1}{2} \left( \int_{0}^{\xi/2} e^{B (\xi-s)} B^{-1} F(s) \, ds + \int_{\xi/2}^{\xi} e^{B (\xi-s)} B^{-1} F(s) \, ds \right) \]

\[ = -\frac{1}{2} \left( V_{21}(\xi) + V_{22}(\xi) \right). \]
One has

\[ \|V_{21}(\xi)\|_E \leq C \left( \frac{\xi/2}{0} e^{-\omega(\xi-s)} ds \right) \|F\|_{h_0^{2\sigma}(\Re)} \leq \frac{C}{\omega} (e^{-\omega \xi/2} - e^{-\omega \xi}) \|F\|_{h_0^{2\sigma}(\Re)}, \]

consequently,

\[ \lim_{\xi \to +\infty} V_{21}(\xi) = 0. \]

Since \( \lim_{\xi \to +\infty} F(\xi) = 0 \), then

\[ \lim_{\xi \to +\infty} \sup_{\frac{s}{2} \leq s \leq \xi} \|F(s)\|_E = 0. \]

So,

\[ \|V_{22}(\xi)\|_E \leq C \sup_{\frac{s}{2} \leq s \leq \xi} \|F(s)\|_E \int_{\xi/2}^{\xi} e^{B(\xi-s)} \|L(\xi)\| ds \]
\[ \leq C \left( 1 - e^{-\omega \xi/2} \right) \left( \sup_{\frac{s}{2} \leq s \leq \xi} \|f(s)\|_E \right). \]

Therefore,

\[ \lim_{\xi \to +\infty} \|V_{22}(\xi)\|_E = 0. \]

In the same way we obtain

\[ \lim_{\xi \to +\infty} V_3(\xi) = 0, \quad \lim_{\xi \to +\infty} V_4(\xi) = 0. \]

We have the following result summarizing the complete analysis of \( V \) for \( F \in h_0^{2\sigma}(\Re) \).

**Proposition 7** Let \( \varphi \in D(A) \). Then \( V \) given in (23) is the unique solution of (20) satisfying

1. \( V'' \), \( AV(.) \in C_b([0, +\infty[; E) \) if and only if \( F(0) - A\varphi = F(0) + B^2 \varphi \in D(A) \),
2. \( V'' \), \( AV(.) \in h_0^{2\sigma}([0, +\infty[; E) \) if and only if \( F(0) - A\varphi = F(0) + B^2 \varphi \in D_A(\sigma) \)
Sketch of the proof. Recall that

\[
B^2V(\xi) = e^{B\xi} (B^2\varphi + F(0)) \\
+ \frac{e^{B\xi}}{2} \left( \int_0^{+\infty} B e^{B s} (F(s) - F(0)) ds \right) \\
- \frac{1}{2} \int_0^{\xi} B e^{B(\xi-s)} (F(s) - F(\xi)) ds \\
- \frac{1}{2} \int_{-\infty}^{+\infty} B e^{B(s-\xi)} (F(s) - F(\xi)) ds \\
+ F(\xi).
\]  

The proof is essentially based on the representation (24) and all the properties proved in [14], in particular see Proposition 1.2, p. 20 and Theorem 4.5, p. 53. For instance, the term

\[ e^{B\xi} (B^2\varphi + F(0)), \]

is continuous at 0 if and only if \( F(0) + B^2\varphi \in \overline{D(A)} \) and its limit when \( \xi \to 0^+ \) is \( B^2\varphi + F(0) \). The second term writes

\[
\frac{e^{B\xi}}{2} \left( \int_0^{+\infty} B e^{B s} (F(s) - F(0)) ds \right) \\
= \frac{e^{B\xi}}{2} \left( \int_0^1 B e^{B s} (F(s) - F(0)) ds + \int_1^{+\infty} B e^{B s} (F(s) - F(0)) ds \right) \\
= \frac{e^{B\xi}}{2} [(a) + (b)],
\]

it is well known that \( (a) \in D_A(\sigma, +\infty) \) when \( F \) is only in \( C^2_b([0, +\infty]; E) \) and \( (a) \in D_A(\sigma) \) in our case \( F \in h^2_b([0, +\infty]; E) \) while \( (b) \) is very regular since it belongs to \( D(B^k) \) for all \( k \in \mathbb{N}^* \). In the same way we analyze the other integrals in \( B^2V(\xi) \).

Remark also that each term in (25) tends to 0 when \( \xi \to +\infty \); the proof is the same as for \( V \).
Going back to our operational problem (14), one obtains obviously, for \( \xi \geq \xi_0 \)

\[
v(\xi) = V(\xi - \xi_0) = e^{B(\xi - \xi_0)} \varphi + \frac{1}{2} \int_{-\infty}^{\xi - \xi_0} e^{B(\xi - \xi_0 - s)} B^{-1} F(s) ds
\]

\[
- \frac{1}{2} \int_{0}^{\xi - \xi_0} e^{B(\xi - \xi_0 - s)} B^{-1} F(s) ds
\]

\[
- \frac{1}{2} \int_{\xi - \xi_0}^{+\infty} e^{B(s - \xi + \xi_0)} B^{-1} F(s) ds,
\]

and the following result

**Proposition 8** Let \( \varphi \in C^2([0, 1]) \) such that \( \varphi(0) = \varphi(1) = 0 \). Then \( v \) given in (26) is the unique solution of (14) satisfying

1. \( \frac{\partial^2 v}{\partial \xi^2} \frac{\partial^2 v}{\partial \eta^2} \in C_b([\xi_0, +\infty[; C([0, 1])) \) if and only if
   \[
   \left\{ \begin{array}{l}
   \eta \mapsto f(\xi_0, \eta) - \varphi''(\eta) \in C([0, 1]) \text{ and } \\
   f(\xi_0, 0) - \varphi''(0) = f(\xi_0, 1) - \varphi''(1) = 0.
   \end{array} \right.
   \]

2. \( \frac{\partial^2 v}{\partial \xi^2} \frac{\partial^2 v}{\partial \eta^2} \in h^2_b([\xi_0, +\infty[; C([0, 1])) \) if and only if
   \[
   \left\{ \begin{array}{l}
   \eta \mapsto f(\xi_0, \eta) - \varphi''(\eta) \in h^2([0, 1]) \text{ and } \\
   f(\xi_0, 0) - \varphi''(0) = f(\xi_0, 1) - \varphi''(1) = 0.
   \end{array} \right.
   \]

**Remark 9** Let \( \varphi = 0 \). We can also give the following representation of the solution by using the Grisvard method, see [5]

\[
V(\xi) = -\frac{1}{2\pi} \int_{\gamma_1} \int_{0}^{\xi} e^{-\sqrt{-\xi} z} \sinh \frac{\sqrt{-\xi} s}{\sqrt{-z}} (A - zI)^{-1} F(s) ds dz
\]

\[
-\frac{1}{2\pi} \int_{\gamma_1} \int_{0}^{+\infty} e^{-\sqrt{-\xi} z} \sinh \frac{\sqrt{-\xi} z}{\sqrt{-z}} (A - zI)^{-1} F(s) ds dz,
\]

where \( \gamma_1 \) is the boundary of

\[
S(\omega_0, \epsilon_0) = \{ z \in \mathbb{C} : |\arg z| \leq \omega_0 \} \cup B(0, \epsilon_0).
\]

Thus

\[
V(\xi, \eta) = V(\xi) = V(\eta)
\]

\[
= -\frac{1}{2\pi} \int_{\gamma_1} \int_{0}^{\xi} e^{-\sqrt{-\xi} z} \sinh \frac{\sqrt{-\xi} s}{\sqrt{-z}} ((A - zI)^{-1} F(s)) (\eta) ds dz
\]

\[
-\frac{1}{2\pi} \int_{\gamma_1} \int_{0}^{+\infty} e^{-\sqrt{-\xi} z} \sinh \frac{\sqrt{-\xi} z}{\sqrt{-z}} ((A - zI)^{-1} F(s)) (\eta) ds dz;
\]
or equivalently
\[ v(\xi, \eta) = v(\xi)(\eta) = V(\xi - \xi_0)(\eta) \]
\[ = -\frac{1}{2i\pi} \int_{\gamma_1}^{\xi-\xi_0} \int_0 e^{-\sqrt{-z}(\xi-\xi_0)} \sinh \sqrt{-z} \left((A - zI)^{-1}F(s)\right)(\eta) dsdz \]
\[ - \frac{1}{2i\pi} \int_{\gamma_1}^{+\infty} \int_{\xi-\xi_0}^{\xi_0} e^{-\sqrt{-z}s} \sinh \sqrt{-z}(\xi - \xi_0) \left((A - zI)^{-1}F(s)\right)(\eta) dsdz; \]

but in our case, one has
\[ \left((A - zI)^{-1}F(s)\right)(\eta) \]
\[ = -\int_0^{\eta} \sinh \sqrt{z}(1 - \eta) \sinh \sqrt{z} \frac{F(s)(\tau)}{\sqrt{z} \sinh \sqrt{z}} d\tau \]
\[ - \int_{\eta}^{1} \sinh \sqrt{z}(1 - \tau) \sinh \eta \sqrt{z} \frac{F(s)(\tau)}{\sqrt{z} \sinh \sqrt{z}} d\tau \]
\[ = \int_0^{1} K_1(\eta, \tau) F(s)(\tau) d\tau \]
\[ = \int_0^{1} K_1(\eta, \tau) f(s + \xi_0, \tau) d\tau, \]

with some natural modification for \((A - zI)^{-1}\) near zero (which is deduced from the explicit calculus of \(A^{-1}\)).

We then obtain the formula
\[ v(\xi, \eta) \]
\[ = -\frac{1}{2i\pi} \int_{\gamma_1}^{\xi-\xi_0} \int_0 e^{-\sqrt{-z}(\xi-\xi_0)} \sinh \sqrt{-z} \left[ \int_0^{1} K_1(\eta, \tau) f(s + \xi_0, \tau) d\tau \right] dsdz \]
\[ - \frac{1}{2i\pi} \int_{\gamma_1}^{+\infty} \int_{\xi-\xi_0}^{\xi_0} e^{-\sqrt{-z}s} \sinh \sqrt{-z}(\xi - \xi_0) \left[ \int_0^{1} K_1(\eta, \tau) f(s + \xi_0, \tau) d\tau \right] dsdz. \]

### 4.2. Optimal results using \(F \in L^\infty([0, +\infty[; h^{2\sigma} ([0, 1]))\)

In this case, we decompose our problem
\[
\begin{align*}
\Delta_{(\xi, 0)} v_{1}(\xi, \eta) &= f_1(\xi, \eta), \quad (\xi, \eta) \in Q_{\xi_0}, \\
v_{1}(\xi_0, \eta) &= \varphi(\eta), \quad 0 < \eta < 1, \\
v_{1}(\xi, 0) &= v_{1}(\xi, 1) = 0, \quad \xi > \xi_0,
\end{align*}
\] (27)

in the two following problems
\[
\begin{align*}
\Delta_{(\xi, 0)} v_{1}(\xi, \eta) &= 0 = f_1(\xi, \eta), \quad (\xi, \eta) \in Q_{\xi_0}, \\
v_{1}(\xi_0, \eta) &= \varphi(\eta), \quad 0 < \eta < 1, \\
v_{1}(\xi, 0) &= v_{1}(\xi, 1) = 0, \quad \xi > \xi_0.
\end{align*}
\] (28)
and
\[
\begin{align*}
\begin{cases}
\Delta (\xi, \eta) v_2 (\xi, \eta) = f (\xi, \eta), & (\xi, \eta) \in Q, \\
v_2 (\xi_0, \eta) = 0, & 0 < \eta < 1, \\
v_2 (\xi, 0) = v_2 (\xi, 1) = 0, & \xi > \xi_0.
\end{cases}
\end{align*}
\] (29)

As above, clearly (28) leads to
\[
\begin{align*}
\begin{cases}
V^{''} (\xi) + A V_1 (\xi) = F_1 (\xi) = 0 & \xi > 0, \\
V_1 (0) = \varphi,
\end{cases}
\end{align*}
\] (30)

and here
\[
F_1 = 0 \in C_b ([0, +\infty[; h^{2\alpha} ([0, 1])]) = C ([0, +\infty[; D_A (\sigma))
\]

since, one has exactly
\[
D_A (\sigma) = \{ \phi \in h^{2\alpha} ([0, 1]) : \phi(0) = \phi(1) = 0 \} = h^{2\alpha}_0 ([0, 1])
\]

Therefore we can use for the abstract equation (30), the same operational techniques as above and due to [14], see Theorem 5.5, p. 60. We then obtain the following result.

**Proposition 10** Let \( \varphi \in D (A) \). Then there exists a unique solution \( V_1 \) to (30) satisfying
\begin{enumerate}
\item \( V^{''}_1, A V_1 (.) \in C_b ([0, +\infty[; E) \)
\item if \( A \varphi \in D_A (\sigma) \) then \( V^{''}_1, A V_1 (.) \in h^{2\alpha}_b ([0, +\infty[; E) \cap L^\infty ([0, +\infty[; h^{2\alpha} ([0, 1])]) \)
\end{enumerate}

The same results hold true for \( v_1 \) which gives

**Proposition 11** Let \( \varphi \in C^2 ([0, 1]) \) such that \( \varphi (0) = \varphi (1) = 0 \). Then there exists a unique solution \( v_1 \) to (28) satisfying
\begin{enumerate}
\item \( \frac{\partial^2 v_1}{\partial \xi^2}, \frac{\partial^2 v_1}{\partial \eta^2} \in C_b ([\xi_0, +\infty[; E) \)
\item if
\[
\eta \mapsto \varphi'' (\eta) \in h^{2\alpha} ([0, 1]) \text{ and } \varphi'' (0) = \varphi'' (1) = 0,
\]
then \( \frac{\partial^2 v_1}{\partial \xi^2}, \frac{\partial^2 v_1}{\partial \eta^2} \in h^{2\alpha}_b ([\xi_0, +\infty[; E) \cap L^\infty ([\xi_0, +\infty[; h^{2\alpha} ([0, 1])]) \)
\end{enumerate}

It remains to analyze (29) with
\[
f \in L^\infty ([\xi_0, +\infty[; h^{2\alpha} ([0, 1])])
\]
without boundary condition on \( f (\xi, .) \) at 0 and 1. One must invert the abstract writing of (29) in order to use the regularity with respect to \( \eta \). We write (29) in the following form
\[
\begin{align*}
\begin{cases}
V^{''}_2 (\eta) + A_2 V_2 (\eta) = F (\eta) & 0 < \eta < 1, \\
V_2 (0) = V_2 (1) = 0,
\end{cases}
\end{align*}
\] (31)
in the space $E_2 = L^\infty([\xi_0, +\infty])$ where for all $\eta \in [0, 1]$ and for a.e. $\xi \in [\xi_0, +\infty]$

$F(\eta) : \eta \mapsto F(\eta)(\xi) = f(\xi, \eta), \quad V_2(\eta) : \xi \mapsto V_2(\eta)(\xi) = v_2(\xi, \eta),$

and $A_2$ is the closed linear operator

$$\left\{ \begin{array}{l}
D(A_2) = \{ \psi \in W^{2, \infty}([\xi_0, +\infty]) : \psi(\xi_0) = 0 \} \\
(A_2\psi)(\xi) = \psi''(\xi).
\end{array} \right.$$ 

Note that here also $D(A_2)$ is not dense in $E = L^\infty([\xi_0, +\infty])$.

As for $A$, operator $A_2$ satisfies the Krein-ellipticity property, that is: $\mathbb{R}^+ \subset \rho(A_2)$ and $\exists C > 0 : \forall \lambda \geq 0$

$$\|(A_2 - \lambda I)^{-1}\|_{L(E)} \leq \frac{C}{1 + |\lambda|},$$

which implies that operator $B_2 = (-A_2)^{1/2}$ is well defined and is the infinitesimal generator of the generalized analytic semigroup $(e^{tB_2})_{t > 0}$.

Therefore the techniques used in the above subsection apply and one obtains

**Proposition 12** There exists a unique solution $V_2$ of (31) satisfying

1. $V''_2, A_2V_2(.) \in C([0, 1]; E_2)$ if and only if $F(0), F(1) \in \overline{D(A_2)}$,
2. $V''_2, AV_2(.) \in h^{2\alpha}([0, 1]; E_2)$ if and only if $F(0), F(1) \in D_{A_2}(\sigma)$.

This Proposition has the equivalent for $v_2$ by inverting the variables $(\eta, \xi)$

**Proposition 13** There exists a unique solution $v_2(\xi, \eta)$ of (27) satisfying

1. $\frac{\partial^2 v_2}{\partial \eta^2}, \frac{\partial^2 v_2}{\partial \xi^2} \in L^\infty([\xi_0, +\infty]; C([0, 1]))$ if and only if

$$\left\{ \begin{array}{l}
\xi \mapsto f(\xi, 0), \ f(\xi, 1) \in C_b([\xi_0, +\infty]) \\
f(\xi_0, 0) = f(\xi_0, 1) = 0
\end{array} \right.$$

2. $\frac{\partial^2 v_2}{\partial \eta^2}, \frac{\partial^2 v_2}{\partial \xi^2} \in L^\infty([\xi_0, +\infty]; h^{2\alpha}([0, 1]))$ if and only if

$$\xi \mapsto f(\xi, 0), \ f(\xi, 1) \in D_{A_2}(\sigma).$$

Therefore we summarize the results in the case of $F \in L^\infty([\xi_0, +\infty]; h^{2\alpha}([0, 1]))$ by writing for $v = v_1 + v_2$

**Proposition 14** Let $\varphi \in C^2([0, 1])$ such that $\varphi(0) = \varphi(1) = 0$. Then there exists a unique solution $v$ such that if

$$\left\{ \begin{array}{l}
\eta \mapsto \varphi''(\eta) \in h^{2\alpha}([0, 1]) \text{ and } \varphi''(0) = \varphi''(1) = 0 \\
\xi \mapsto f(\xi, 0), \ f(\xi, 1) \in D_{A_2}(\sigma),
\end{array} \right.$$

then $\frac{\partial^2 v}{\partial \xi^2}, \frac{\partial^2 v}{\partial \eta^2} \in L^\infty([\xi_0, +\infty]; h^{2\alpha}([0, 1]))$. 

44
4.3. Complete results using $f \in h^{2\sigma}(\xi_0)$

Now we are in position to summarize all the results concerning our Problem (12) with $f \in h^{2\sigma}_b([0, \infty]; E) \cap L_\infty([\xi_0, \infty]; h^{2\sigma}([0, 1])) = h^{2\sigma}_b(\xi_0)$.

From the results in the above subsections, one obtains

**Proposition 15** Let $\varphi \in C^2([0, 1])$ such that $\varphi(0) = \varphi(1) = 0$. Then there exists a unique solution $v$ of (10) such that

1. if
   \[
   \begin{cases}
   \eta \mapsto \varphi''(\eta) \in h^{2\sigma}([0, 1]) \text{ and } \varphi''(0) = \varphi''(1) = 0 \text{ and} \\
   \xi \mapsto f(\xi, 0), \ f(\xi, 1) \in D_{A_2}(\sigma),
   \end{cases}
   \]
   then $\frac{\partial^2 v}{\partial \xi^2}, \frac{\partial^2 v}{\partial \eta^2} \in L_\infty([\xi_0, \infty]; h^{2\sigma}([0, 1]))$.

2. $\frac{\partial^2 v}{\partial \xi^2}, \frac{\partial^2 v}{\partial \eta^2} \in h^{2\sigma}_b([\xi_0, \infty]; C([0, 1]))$ if and only if
   \[
   \begin{cases}
   \eta \mapsto f(\xi_0, \eta) - \varphi''(\eta) \in h^{2\sigma}([0, 1]) \text{ and} \\
   f(\xi_0, 0) - \varphi''(0) = f(\xi_0, 1) - \varphi''(1) = 0.
   \end{cases}
   \]

Let us focus ourselves on the case when $\varphi = 0$,

recall that the conditions

$\xi \mapsto f(\xi, 0), \ f(\xi, 1) \in D_{A_2}(\sigma),$ \hspace{1cm} (32)

mean that

$f(\xi_0, 0) = f(\xi_0, 1) = 0,$

then, one has

**Proposition 16** Let $f \in h^{2\sigma}(\xi_0)$, $(\sigma \in ]0, 1/2[)$ such that

$\xi \mapsto f(\xi_0, 0) = f(\xi_0, 1) = 0.$ \hspace{1cm} (33)

Then there exists a unique solution $v$ of (12) such that

$\frac{\partial^2 v}{\partial \xi^2}, \frac{\partial^2 v}{\partial \eta^2} \in L_\infty([\xi_0, \infty]; h^{2\sigma}([0, 1])) \cap h^{2\sigma}_b([\xi_0, \infty]; C([0, 1])) = h^{2\sigma}_b(\xi_0).$
Set
\[ X_d = \{ v \in h^{2+2\sigma}(Q_{\xi_0}) : v = 0 \text{ on } \partial Q_{\xi_0} \} \]  
(34)

and
\[ X_a = \{ f \in h_b^{2\sigma}(Q_{\xi_0}) : f(\xi_0,0) = f(\xi_0,01) = 0 \} \]  
(35)

Therefore, we deduce that the Laplace operator
\[ \Delta : X_d \rightarrow X_a, \]

is an isomorphism.

At this level, we recall that we look to study the regularity of the solution of Problem (6) at the vicinity of \( D_\infty \) (given by 4). For this reason, we introduce the two following operators

\[ T : h_b^{2\sigma}(Q_{\xi_0}) \rightarrow h_b^{2\sigma}(Q_{\xi_0}) \]
\[ f \quad \mapsto \quad k(\xi) f, \]

where \( k : \mathbb{R}^+ \rightarrow \mathbb{R} \) is the the truncation function defind by

\[
\begin{align*}
  k(\xi) &= 0 & 0 \leq \xi \leq 2\xi_0 := \xi_1, \\
  k(\xi) &= \xi - \xi_1 & \xi_1 \leq \xi \leq 2\xi_1 := \xi_2, \\
  k(\xi) &= 1 & \xi \geq \xi_2.
\end{align*}
\]

\[ \bar{L} : h_b^{2+2\sigma}(Q_{\xi_0}) \rightarrow h_b^{2\sigma}(Q_{\xi_0}) \]
\[ f \quad \mapsto \quad (T \circ L) f \]

where \( L \) is given by (8).

**Lemma 17** Let \( \sigma \in ]0,1/2[ \). One has

1. The linear operator \( T \) is continuous with
\[ \|Tf\|_{h^{2\sigma}(Q_{\xi_0})} \leq 2 \|f\|_{h^{2\sigma}(Q_{\xi_0})}. \]

2. The linear operator \( \bar{L} \) is continuous with
\[ \|\bar{L}f\|_{h_b^{2+2\sigma}(Q_{\xi_0})} \leq 2 \|L\|_{h_b^{2+2\sigma}(Q_{\xi_0})}. \]

Using the same argument as in [7] and Keeping in mind the results of the previous section. We deduce that there exist \( \xi^* \geq \xi_2 \) large enough such that
\[ \Delta + \bar{L} : X_d \rightarrow X_a \]

is an isomorphism. This justifies our main result concerning our complete transformed problem (6)
Proposition 18 Let $f \in h_b^{2\sigma}(Q_{\xi_0})$, $(\sigma \in ]0, 1/2[)$ such that 
\[ \xi \mapsto f(\xi_0, 0) = f(\xi_0, 1) = 0. \]
Then there exists $\xi^* \geq \xi_2$ such that (6) admits a unique strict solution $v$ satisfying 
\[ \frac{\partial^2 v}{\partial \xi^2}, \frac{\partial^2 v}{\partial \eta^2} \in h_b^{2\sigma}(Q_{\xi^*}). \]

Remark 19 In the sequel, for simplicity, we assume that $\xi^* = \xi_2$.

4.4. Go back to the original problem

Let $\pi$ the unique variationnal solution of Problem (1), see [7]. Set 
\[ u = \Theta(x) \pi \]
where $\Theta(x) \in C^\infty([0, x_0])$ such that 
\[
\begin{cases}
0 \leq \Theta \leq 1 \\
\Theta(x) = 0 \quad x > x_1 := \Pi^{-1}(\xi_1) \\
\Theta(x) = 1 \quad x \leq x_2 := \Pi^{-1}(\xi_2)
\end{cases}
\]

Remark 20 It is easy to see that:

1. \[
\begin{cases}
u = 0 \quad x > x_1 \\
\nu = \pi \quad 0 < x < x_2
\end{cases}
\]
2. $\Delta u \in h^{2\sigma}(\Omega_{x_0})$.

Taking into account, the results of the previous section, we conclude that for $x \leq x_2$, one has 
\[ \pi = \Pi^{-1}(v) \]
where $v$ is the unique solution of (6).

Now, we give the following result describing the effect of the inverse change of variables.

Lemma 21 Let $\sigma \in ]0, 1/2[$. Then 
\[ g \in h_b^{2\sigma}(Q_{\xi_0}) \Rightarrow (\psi(x))^2\sigma \ h \in h^{2\sigma}(\Omega_{x_0}). \]
Recall that for all $(x, y) \in \Omega$ 
\[ h(x, y) = g \left( \theta^{-1}(x), \frac{y}{\psi(x)} \right). \]
Given a small $\delta > 0$. Let $(x_2, y_2), (x_1, y_1) \in \Omega_{x_0}$ such that 
$(x_2, y_2) \neq (x_1, y_1)$,
and \( \| (x_2 - x_1, y_2 - y_1) \| \leq \delta \). Assume, for instance that \( x_1 \leq x_2, y_1 \leq y_2 \).

First, it is easy to see that

\[
\sup_{(x,y) \in B_1} \left| (\psi(x))^{2\sigma} h(x,y) \right| < \infty.
\]

One has

\[
(\psi(x_2))^{2\sigma} h(x_2, y_2) - (\psi(x_1))^{2\sigma} h(x_1, y_1) = ((\psi(x_2))^{2\sigma} - (\psi(x_1))^{2\sigma}) h(x_2, y_2) + (\psi(x_1))^{2\sigma} (h(x_2, y_2) - h(x_1, y_1))
\]

\[
= P_1 + P_2.
\]

Then

\[
\lim \sup_{\delta \to 0} \frac{|P_1|}{\| (x_2 - x_1, y_2 - y_1) \|^{2\sigma}} = 0.
\]

For \( P_2 \), one has

\[
\frac{|P_2|}{\| (x_2 - x_1, y_2 - y_1) \|^{2\sigma}} = (\psi(x_1))^{2\sigma} \frac{|g \circ \Pi(x_2, y_2) - g \circ \Pi(x_1, y_1)|}{\| \Pi(x_2, y_2) - \Pi(x_1, y_1) \|^{2\sigma}} \| \Pi(x_2, y_2) - \Pi(x_1, y_1) \|^{2\sigma}.
\]

From \( g \in h^{2\sigma}(Q_{\varepsilon_0}) \), we get

\[
\lim_{\delta \to 0} \sup_{\Pi(x_2, y_2) - \Pi(x_1, y_1) \leq \delta} \frac{|g \circ \Pi(x_2, y_2) - g \circ \Pi(x_1, y_1)|}{\| \Pi(x_2, y_2) - \Pi(x_1, y_1) \|^{2\sigma}} = 0.
\]

It remains to estimate the second fraction; one has

\[
\| \Pi(x_2, y_2) - \Pi(x_1, y_1) \|^{2\sigma} \leq \left\| \left( \theta^{-1}(x_2) - \theta^{-1}(x_1), \frac{y_2}{\psi(x_2)} \frac{y_1}{\psi(x_1)} \right) \right\|^{2\sigma} \leq \left\| \left( \theta^{-1}(x_2) - \theta^{-1}(x_1), \frac{y_2 \psi(x_1) - y_1 \psi(x_2)}{\psi(x_2) \psi(x_1)} \right) \right\|^{2\sigma} \leq \left\| \left( \theta^{-1}(x_2) - \theta^{-1}(x_1), y_2 \frac{\psi(x_1) - \psi(x_2)}{\psi(x_2) \psi(x_1)} + \psi(x_2) \frac{(y_2 - y_1)}{\psi(x_2) \psi(x_1)} \right) \right\|^{2\sigma}
\]

since \( x_1 < x_2 \). Thus

\[
\| \Pi(x_2, y_2) - \Pi(x_1, y_1) \|^{2\sigma} \leq \frac{C}{(\psi(x_1))^{2\sigma}} \| (x_2 - x_1, y_2 - y_1) \|^{2\sigma}
\]

from which we deduce that

\[
\lim_{\delta \to 0} \sup_{\| (x_2 - x_1, y_2 - y_1) \| \leq \delta} \frac{|P_2|}{\| (x_2 - x_1, y_2 - y_1) \|^{2\sigma}} = 0.
\]
Summing up, we get

$$\lim_{\delta \to 0} \sup_{\| (x_2 - x_1, y_2 - y_1) \| \leq \delta} \| (x_2 - x_1, y_2 - y_1) \|^{2\sigma} = 0.$$ 

Now, taking into account this result and using the same techniques as in [7], we give our main result

**Theorem 22** Let $h \in h^{2\sigma}(\Omega)$, $(\sigma \in [0, 1/2])$ satisfying (3). Then, there exists $x_2 < x_0$ such that Problem (1)-(2) admits a unique strict solution $u$ satisfying

$$(\psi(x))^{2\sigma} \partial_y^2 u \text{ and } (\psi(x))^{2\sigma} \partial_x^2 u \in h^{2\sigma}(\overline{\Omega_{x_2}}).$$

where

$$\Omega_{x_2} := \{ (x, y) \in \mathbb{R}^2 : 0 < x < x_2, \ 0 < y < \psi(x) \}. \quad (38)$$

**REFERENCES**


